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Membrane and Bending Stresses in Pressurized Elliptical Cylinders:

Analytical Solutions and Finite Element Models

A Technical Report
by
Charles D. Kuhl

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of
MASTER OF ENGINEERING

August 1996

Major Subject: Civil Engineering

ABSTRACT

This report addresses the analysis of stresses in a thin walled elliptical cylinder using both classical membrane theory, bending theory, and Finite Element Analysis. Numerous analytical solutions from six authors are presented and examined to determine the hoop stresses, bending stresses, and total stresses that arise when elliptical cargo tanks are subjected to uniform internal pressure. Two Finite Element Models were created and evaluated using quadratic beam and shell elements. All resulting stresses are analyzed and compared for accuracy. The final product is an analytical solution procedure that can be used by practicing engineers for elliptical cargo tank design.

Keywords

Beam, Bending, Boundary Condition, Cargo, Circle, Cylinder, Deflection, Deformation, Ellipse, Eccentricity, Elliptical Integral, Finite Element, Finite Element Analysis, Finite Element Method, Force, Hoop, Longitudinal, Membrane, Moment, Pressure, Radial, Shear, Shell, Stress, Tank, Tension

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1. Introduction

1.1 General

Highway cargo tankers are an essential part of the economic infrastructure of the United States, and many other countries throughout the world. They serve as a means of transportation for both hazardous and benign liquids and gases. The products carried in these tankers range from toxic waste and gases to milk and water. Without their use consumers would not have access to many of the goods that are part of every day living.

Finding safer and more economical ways to transport materials is a chief concern of both the Federal Highway Administration (FHWA) and the cargo tanker manufacturing industry. One way to make a tanker safer is to lower the center of gravity. The method most commonly used is to construct the tanks with an elliptical profile instead of a circular profile as shown in Figure 1. This allows the same ground clearance while having a lower center of gravity, and the same or greater cargo capacity. While the elliptical profile does create a safer tanker with regard to over-turning, it creates forces in the shell wall that are not present in simple circular sections. The contents of the tank may also be pressurized which compounds the forces created by weight of the cargo alone.

FIG. 1. Typical Truck Mounted Elliptical Cargo Tank

1.2 Objectives of the Study

The primary objective of this study is to find closed form solutions that accurately predict the bending moments, hoop forces, and bending stresses in pressurized elliptical cargo tanks subjected to uniform internal pressure. The cargo tank evaluated in this report has a major axis of 78.8 in., a minor axis of 47.5 in., is constructed of steel 0.25 in. thick, and has a uniform internal pressure of 10 psi. An attempt to validate the closed form solutions will be done by comparing the results of various solution techniques. The inputs needed to find the bending stresses are the hoop forces and the bending moments that are generated in the shell. The equation that will be used to determine the total stress when small deflections occur is:

$$\sigma_\phi = \left(\frac{N_\phi}{t}\right) + \left(\frac{6M_\phi}{t^2}\right). \tag{1}$$

where N is the hoop force, M is the bending moment, and t is the shell thickness.

There are three major phases in the construction of this study. First, development of formulae using pure membrane theory. Second, construction of formulae using bending theory. Finally, comparison of the various closed form solutions with one another and against the results of a finite element model analysis.

2. FLUGGE MEMBRANE THEORY

2.1 General

A cylinder is created by moving a straight line perpendicularly along a simple closed curve. Thus, through every point on the surface of the cylinder there is a line that lies on the surface. These lines are called generators. When a plane that is normal the generators intersects the cylinder, a profile is formed. The profiles are identical for all generators. Cylinders are titled after the shape of the profile. In this case, the profile considered is an ellipse, thus, this is a study of elliptical cylinders.

2.2 Coordinate Relations

The length of the cylinder is l, and an arbitrary profile is chosen as a reference point from which to measure the distance between two selected profiles. The distance x is measured along the generators and is the first of the primary coordinates. The second primary coordinate is phi (ϕ) which is the angle between a surface tangent and the horizontal. The dimensions of the semi-major and semi-minor axes will define the geometry of the profile, and the x-y coordinates defined any point on the cylinder surface by using the angle phi. The radial direction is orthogonal to the tangent at any point on the surface of the cylinder.

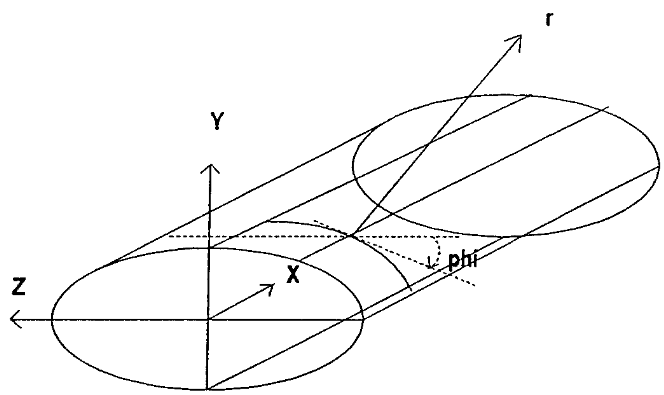

FIG. 2. Coordinates of a Cylinder

To analyze the cylinder it is necessary to consider an infinitesimally small element on the surface of the shell. The element is bounded in the x direction by x and $x + dx$, and in the ϕ direction by ϕ and $\phi + d\phi$. Membrane forces which act on the element's edges are tangent to the middle surface. The stresses per unit length of the section are the normal and shear stresses. Normal stresses are expressed by the components N_x and N_ϕ, and shear stresses are $N_{x\phi} = N_{\phi x}$. The loads per unit area are comprised of the elements p_x and p_ϕ in the directions of increasing x and ϕ with a radial element p_r positive in the outward direction (Fig. 3).

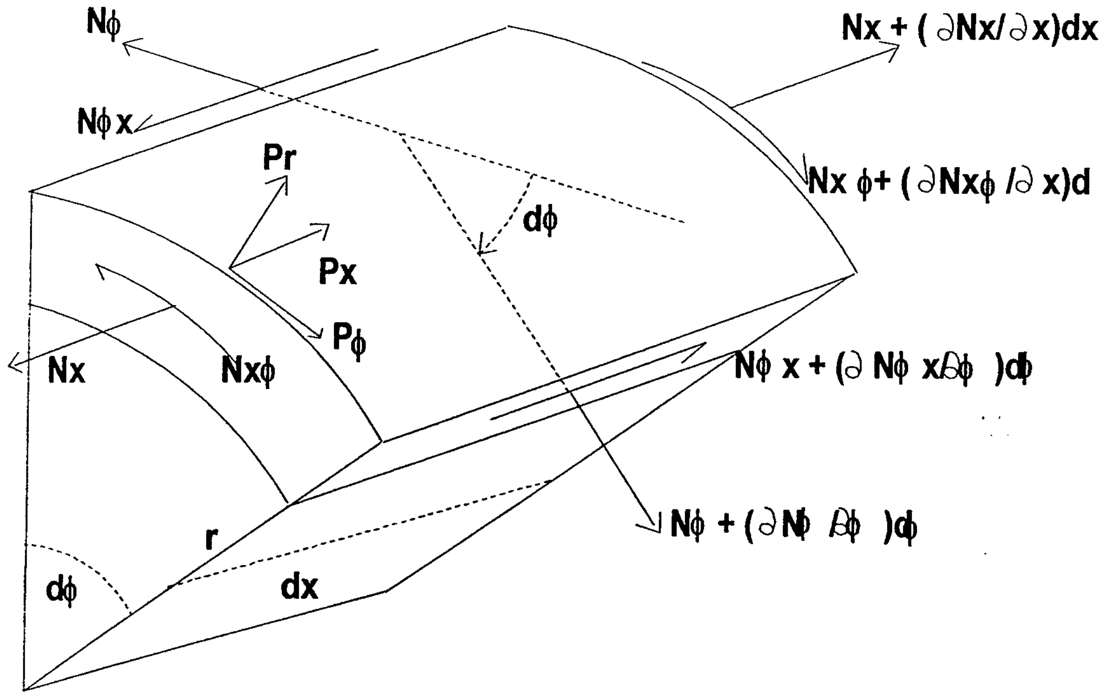

FIG. 3. Membrane Forces and Stresses on a Surface Element

2.3 Equation Development

Equilibrium of the shell element in figure 2 results in the following differential equations for the membrane stresses:

Equilibrium in the x-direction gives:

$$\frac{\partial N_x}{\partial x} dx \cdot rd\phi + \frac{\partial N_{\phi x}}{\partial \phi} d\phi \cdot dx + p_x \cdot dx \cdot rd\phi = 0 \qquad (2)$$

Summation of forces parrallel to the tangent yields:

$$\frac{\partial N_\phi}{\partial \phi} d\phi \cdot dx + \frac{\partial N_{x\phi}}{\partial x} dx \cdot rd\phi + p_\phi \cdot dx \cdot rd\phi = 0 \qquad (3)$$

Equilibrium of forces normal to the surface is:

$$N_\phi dx \cdot d\phi - p_r \cdot dx \cdot rd\phi = 0 \qquad (4)$$

Division by the two differentials gives the differential equations for the membrane forces experienced by the shell.

$$\frac{\partial Nx\phi}{dx} = -p_\phi - \frac{1}{r}\frac{\partial N_\phi}{\partial \phi} \qquad (5)$$

$$\frac{\partial N_x}{\partial x} = -p_x - \frac{1}{r}\frac{\partial N\phi x}{d\phi} \qquad (6)$$

$$N_\phi = p_r r \qquad (7)$$

It becomes evident from equation 4 that the hoop force (N_ϕ) is entirely a function of the internal pressure and the radius of curvature.

The boundary conditions of primary concern for this problem are those at the end supports. Support is supplied in the radial direction by a ring, truss, or thin solid wall normally in the form of a baffle or head. For an elliptical cylinder tank the support will most likely be a stiffener ring or a thin plate head depending on the manufacturer. The end support will be defined as a diaphragm in this report.

The simplest and most important boundary conditions of membrane theory for cylinders coincides with the conditions of this study. The cylinder is of length *l* supported by diaphragms at both ends. The distribution, along the length, of the shearing force $Nx\phi$ is the same as that of a simply supported beam which carries a uniformly distributed load. The Nx forces are distributed in the *x*-direction similar to the bending moments of a beam. Thus, a cylindrical shell behaves like a simply supported beam and transfers all the shearing forces $Nx\phi$ to the diaphragms at the ends of the span. However, due to the geometry of the cross section of a cylinder the distribution of the forces Nx and $Nx\phi$ cannot be derived from the simple beam application.

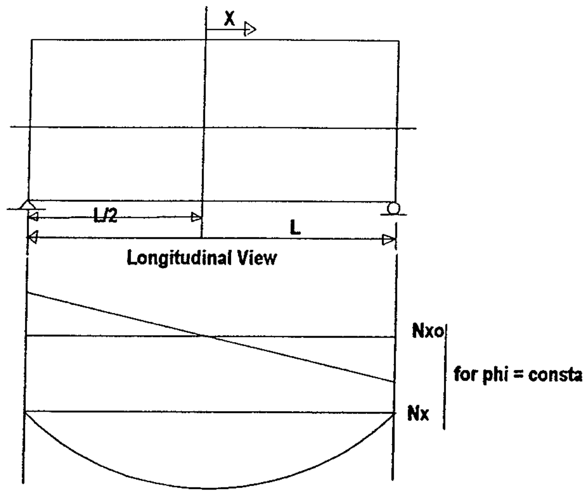

FIG. 4. Cylindrical Shell Supported By Diaphragms at Ends

When the coordinate x is considered at the mid-point of the longitudinal span, the following boundary conditions are found:

$$N_x \equiv 0 \quad \text{at} \quad x = \pm l/2 \tag{8}$$

Using N_x from equation 4b, f_1 and f_2 are defined as:

$$f_1(\phi) \equiv 0, \text{ and } f_2(\phi) = -\frac{l^2}{8r}\frac{dF(\phi)}{d\phi} \tag{9}$$

Thus, the following two general equations for the membrane stresses in cylinders are found. When equation 7 is added to this pair the generalized set of equations is complete.

$$N_{x\phi} = -xF(\phi) \tag{10}$$

$$N_x = \frac{1}{8r}\left(l^2 - 4x^2\right)\frac{dF(\phi)}{d\phi} \tag{11}$$

$$\mathrm{N}_\phi = p_r r \tag{7}$$

2.4 Circular Cylinder

A brief study of the circular cylinder will be beneficial prior to progressing the more complex elliptical cylinder.

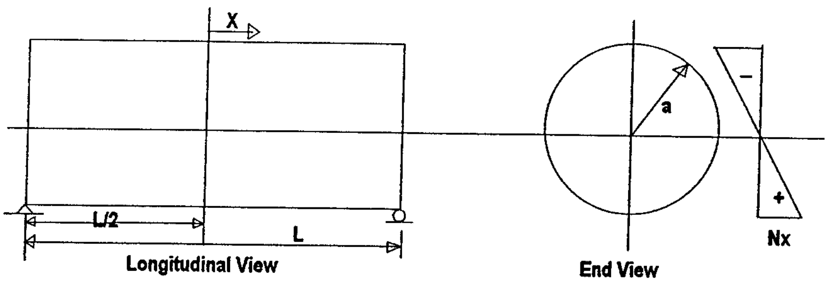

FIG. 5. Circular Cylinder

This serves as the simplest case of a cylinder with r = a is a constant. If the cylinder is filled with a fluid with specific gravity (gamma) and an internal pressure p_o the external forces encountered are $p_x = p_\phi = 0$ $p_r = p_o - \gamma a\cos\phi$. Applying the boundary conditions of Fig. 4 to equations 10, 11, and 7 the specific stress resultants for the circular cylinder are

$$N_\phi = p_o a - \gamma a^2 \cos\phi \tag{12}$$

$$N_{x\phi} = -\gamma a x \sin\phi \tag{13}$$

$$N_x = -\frac{1}{8}\gamma\left(l^2 - 4x^2\right)\cos\phi \tag{14}$$

The internal pressure produces only hoop stresses while the weight of the fluid carried internally causes a bending in the x-direction of the cylinder between the diaphragms. As described previously the shear and normal forces have the same distribution as that of a simply supported beam. The distribution of Nx along the profile is shown in Fig. 5.

2.5 Elliptical Cylinder

The elliptical cylinder studied in this report is shown in figure 6. The cylinder is filled with a liquid and is subject to an internal gas pressure of 10 psi.

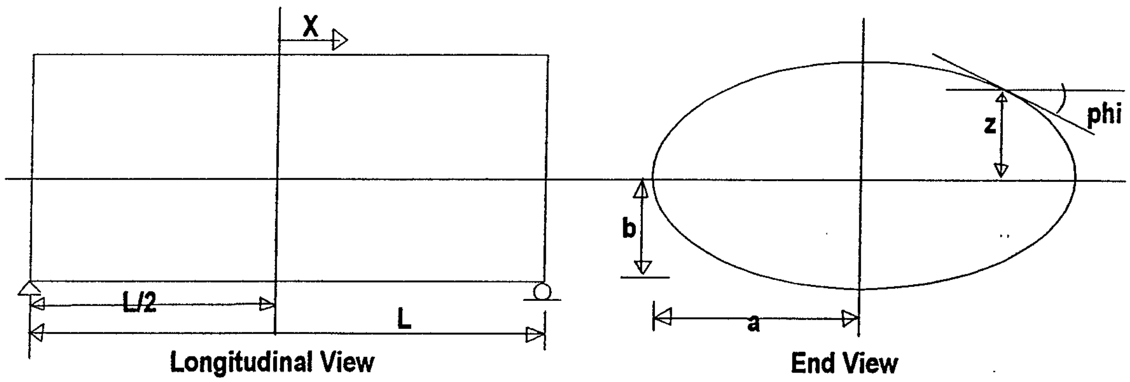

FIG. 6. Elliptical Cylinder

The following equations are geometry specific to the elliptical cylinder and were derived by Wilhelm Flugge (1960) by substituting the more complex conditions of the ellipse into equations 8, 9, and 10. Using the equation of the ellipse the following relations for *z* and *y* are derived which relates *z* and *y* to the phi coordinate:

$$z = \frac{b^2 \cos\phi}{(a^2 \sin^2\phi + b\cos^2\phi)^{1/2}} \tag{15}$$

$$y = \frac{a^2 \sin\phi}{(a^2 \sin^2\phi + b\cos^2\phi)^{1/2}} \tag{16}$$

The radius of curvature r is also derived using the geometry of the ellipse.

$$r = \frac{a^2 b^2}{\left(a^2 \sin^2\phi + b^2\cos^2\phi\right)^{3/2}} \tag{17}$$

By substituting the above equations into equations 5, and 7 the following equations are arrived at for the hoop force:

$$N_\phi = \frac{p_o a^2 b^2}{(a^2 \sin^2\phi + b\cos^2\phi)^{3/2}} - \frac{\gamma a^2 b^4 \cos\phi}{(a^2 \sin^2\phi + b^2\cos^2\phi)^2} \tag{18}$$

Shear force:

$$N_{x\phi} = 3p_o(a^2 - b^2)x\frac{\cos\phi\sin\phi}{a^2\sin^2\phi + b^2\cos^2\phi} - \gamma^2 bx\frac{3(a^2-b^2)\cos\phi + a^2}{(a^2\sin^2\phi + b^2\cos^2\phi)^{3/2}}\sin\phi \qquad (19)$$

Normal force in the x direction:

$$N_x = -\frac{3}{8}p_o\frac{a^2-b^2}{a^2b^2}(l^2 - 4x^2)\frac{a^2\sin^4\phi - b^2\cos^4\phi}{(a^2\sin^2\phi + b^2\cos^2\phi)^{1/2}} +$$

$$\frac{1}{8}\frac{\gamma}{a^2}(l^2 - 4x^2)\frac{8a^4\sin^2\phi - a^2b^2(4+5\sin^2\phi) + 3b^4\cos^2\phi}{a^2\sin^2\phi + b^2\cos^2\phi}\cos\phi \qquad (20)$$

2.6 Flugge Results

These equations are applied to an elliptical cylinder with the following characteristics:

- length span, l = 240 in.
- longitudinal distance, x = 0.1 in.
- horizontal radius, a = 39.4 in.
- vertical radius, b = 23.75 in.
- internal pressure, p_o = 10 psi
- gamma of water = 62.4 pcf or 0.03611 pci

The graphs using these properties are shown in figures 7, and 8. As discussed previously the distribution of Nx and $Nx\phi$ are the same as the bending moment and shearing force of a simply supported beam, and $N\phi$ does not depend on location along the span. The constant pressure produces additional forces in Nx and $Nx\phi$. These are the forces that enable the shell to withstand the applied load without bending stresses according to membrane theory. The diaphragms are needed to take the shearing forces $Nx\phi$ that result from this system. The graphs in figure 6 show the results of the cylinder

filled to the top without any additional internal pressure. These results were obtained by setting the pressure p_o equal to zero in equations 18- 20.

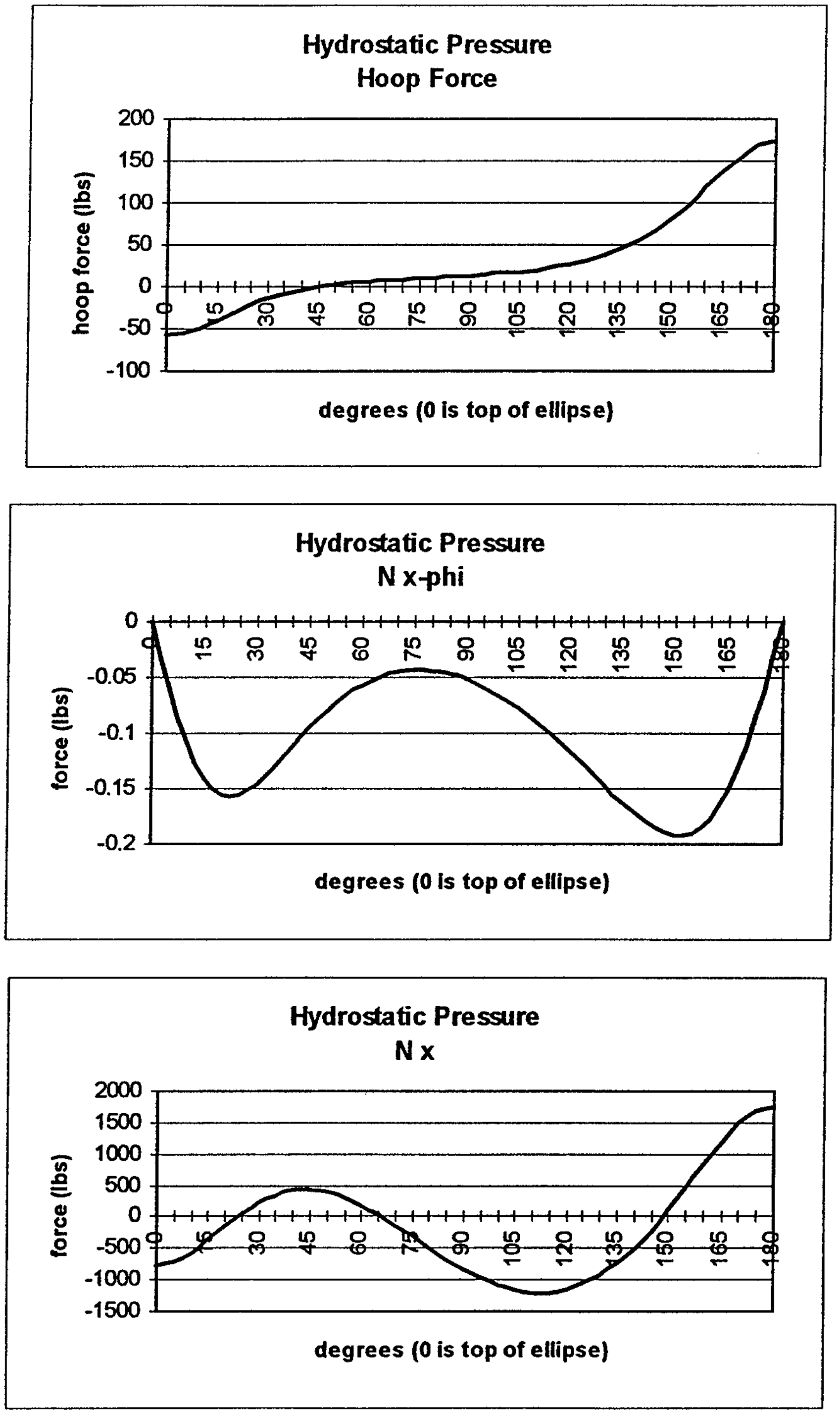

FIG. 7. Flugge Fluid pressure only

Figure 8 displays the results obtained when only internal pressure is considered by setting liquid weight equal to zero in equations 18 - 20.

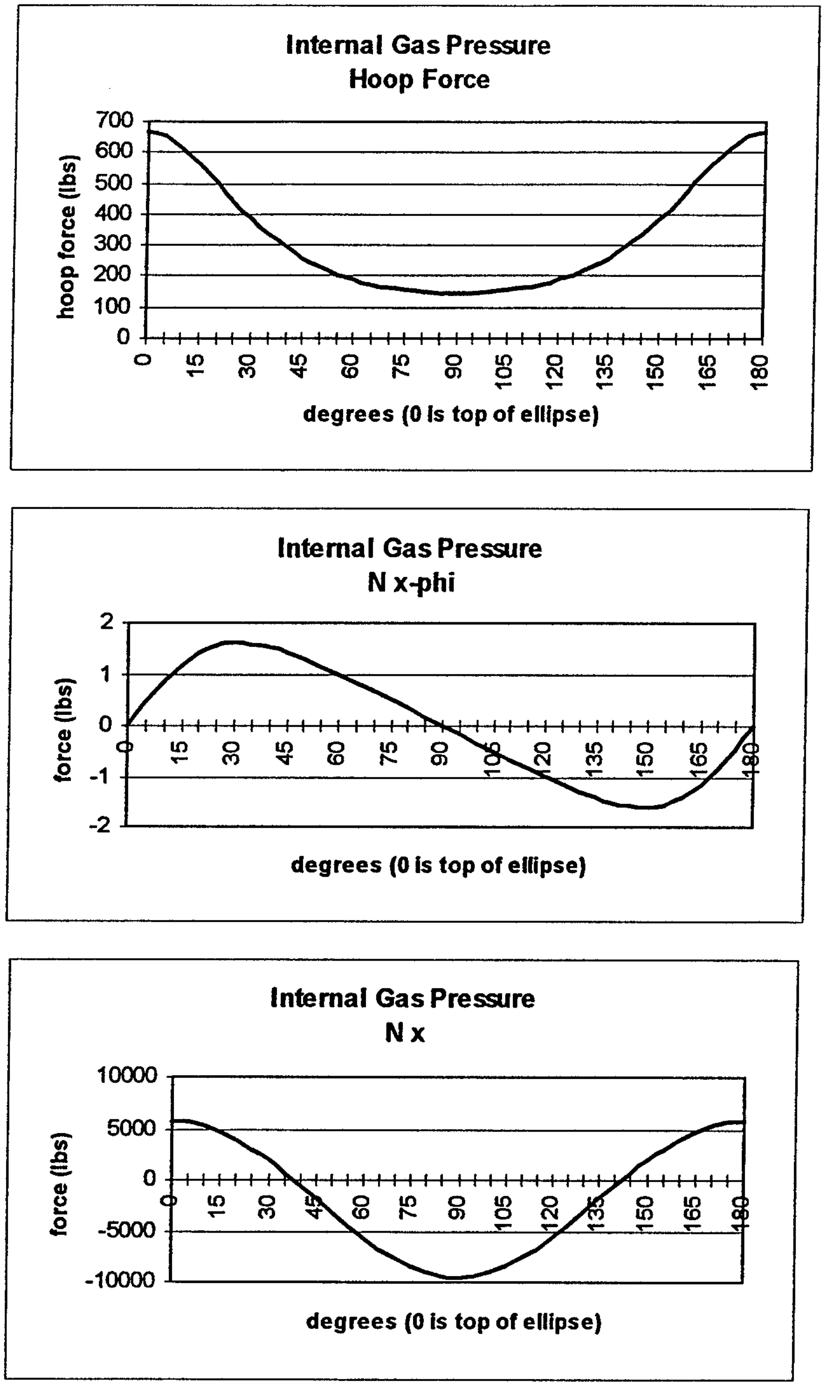

FIG. 8. Flugge Internal Gas Pressure

3. BENDING THEORIES

3.1 KOZIK BENDING THEORY

3.1.1 Equation Development

The equations used in this section were developed by Dr. Thomas J. Kozik, Professor of Mechanical Engineering, at Texas A&M University. The following equations are geometry specific to the elliptical cylinder, just as the previous equations were, which were developed by Flugge. The critical difference is that these equations derived by Kozik account for the bending stresses that were ignored by membrane theory.

3.1.2 Coordinate Relations

Figures 9 through 12 are the coordinate relations used in the development of the equations for bending theory.

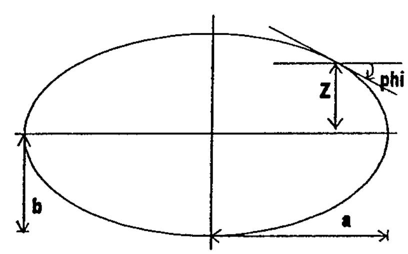

FIG. 9. Profile Coordinate Relations

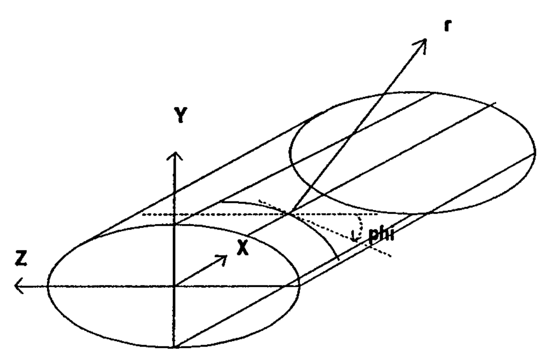

FIG. 10. Elliptical Cylinder Coordinate Relations

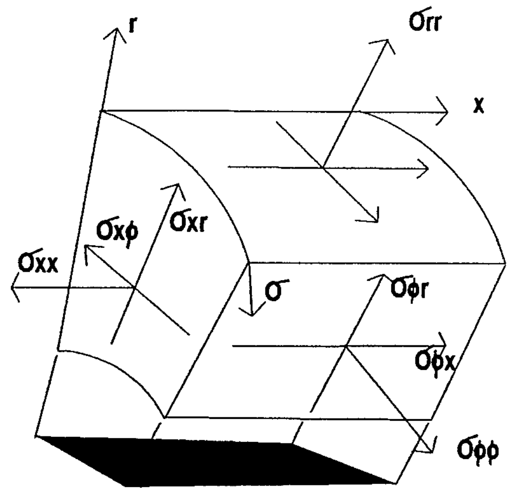

FIG. 11. Stress Block

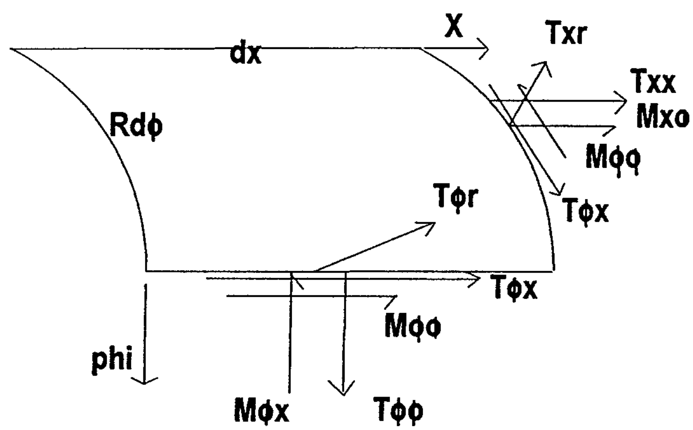

FIG. 12. Stress Resultant Convention

3.1.3 Equations

The strain-displacement, constitutive, compatibility, and equilibrium equations presented below are the culmination of the derivations by Dr. Kozik. As stated previously, they are all geometry specific to the elliptical cylinder.

Strain - Displacement Equations

$$e_{\phi\phi o} = \frac{1}{R}\frac{\partial u_o}{\partial \phi} + \frac{w_o}{R} \tag{21}$$

$$e_{xxo} = \frac{\partial v_o}{\partial x} \tag{22}$$

$$e_{x\phi o} = \frac{\partial u_o}{\partial x} + \frac{1}{R}\frac{\partial v_o}{\partial \phi} \tag{23}$$

$$\omega_o = \frac{1}{2R}\frac{\partial v_o}{\partial \phi} - \frac{1}{2}\frac{\partial u_o}{\partial x} \tag{24}$$

$$\mathrm{K}_\phi = \frac{1}{R^2}\frac{\partial^2 w_o}{\partial \phi^2} + \frac{1}{R^3}\frac{\partial R}{\partial \phi}\frac{\partial w_o}{\partial \phi} + \frac{1}{R}\frac{\partial}{\partial \phi}\frac{u_o}{R} \tag{25}$$

$$\mathrm{K}_x = - \frac{\partial^2 w_o}{\partial x^2} \tag{26}$$

$$\tau = \frac{2}{R}\frac{\partial^2 w_o}{\partial x \partial \phi} + \frac{1}{R}\frac{\partial u_o}{\partial x} \tag{27}$$

$$\mathrm{R} = \frac{a^2 b^2}{\left(a^2 \sin^2\phi + b^2 \cos^2\phi\right)^{3/2}} \tag{28}$$

Constitutive Equations

$$\mathrm{T}_{\phi\phi} = \frac{Eh}{1-\upsilon^2}\,(\mathrm{e}_{\phi\phi o} + \upsilon\,\mathrm{e}_{xxo}) \tag{29}$$

$$\mathrm{T}_{\phi x} = \frac{Eh}{2(1+\upsilon)}\,(\mathrm{e}_{x\phi o}) \tag{30}$$

$$\mathrm{T}_{xx} = \frac{Eh}{1-\upsilon^2}\,(\mathrm{e}_{xxo} + \upsilon\,\mathrm{e}_{\phi\phi o}) + \frac{Eh^3}{12(1-\upsilon^2)\,R}\,(\mathrm{K}_x + \upsilon \mathrm{K}_\phi) \tag{31}$$

$$\mathrm{T}_{x\phi} = \frac{Eh}{2(1+\upsilon)}\,(\mathrm{e}_{x\phi o}) + \frac{Eh^3\tau}{24(1+\upsilon)R} \tag{32}$$

$$\mathrm{M}_{\phi\phi} = \frac{Eh^3}{12(1-\upsilon^{2)}}\,(\mathrm{K}_\phi + \upsilon \mathrm{K}_x) \tag{33}$$

$$\mathrm{M}_{\phi x} = \frac{Eh^3\tau}{24(1+\upsilon)} \tag{34}$$

$$M_{xx} = \frac{Eh^3}{12(1-\upsilon^{2)}}\,(K_x + \upsilon K_\phi) + \frac{Eh^3}{12(1-\upsilon^2)R}\,(e_{xxo} + \upsilon e_{\phi\phi o}) \tag{35}$$

$$M_{x\phi} = \frac{Eh^3\tau}{24(1+\upsilon)} + \frac{Eh^3}{24(1+\upsilon)R}(e_{x\phi o}) \tag{36}$$

Compatibility Equations

$$0 = R\frac{\partial K_\phi}{\partial x} - \frac{1}{2}\frac{\partial \tau}{\partial \phi} - \frac{1}{4R}\frac{\partial e_{\phi xo}}{\partial \phi} + \frac{1}{2R}\frac{\partial \omega_o}{\partial \phi} - \frac{1}{2R^2}\frac{\partial R}{\partial \phi}\left(\omega_o - \frac{e_{\phi xo}}{2}\right) \tag{37}$$

$$0 = \frac{R}{2}\frac{\partial \tau}{\partial x} - \frac{1}{R}\frac{\partial e_{xxo}}{\partial \phi} + \frac{3}{4}\frac{\partial e_{x\phi o}}{\partial x} + \frac{1}{2}\frac{\partial \omega_o}{\partial x} \tag{38}$$

$$0 = \frac{1}{R}K_x + \frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{1}{R}\frac{e_{xxo}}{\partial \phi} - \frac{1}{2}\frac{\partial e_{\phi xo}}{\partial x}\right) + \frac{1}{R}\frac{\partial}{\partial x}\left(\frac{R\,e_{\phi\phi o}}{\partial x} - \frac{1}{2}\frac{\partial e_{\phi xo}}{\partial \phi}\right) \tag{39}$$

Equilibrium Equations

$$0 = \frac{1}{R}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{\phi x}}{\partial x} + \frac{1}{R^2}\frac{\partial M_{\phi x}}{\partial \phi} + \frac{1}{R}\frac{\partial M_{\phi x}}{\partial x} + \frac{1}{R}\frac{\partial M_{xx}}{\partial x} + P_\phi \tag{40}$$

$$0 = \frac{\partial T_{xx}}{\partial x} + \frac{1}{R}\frac{\partial T_{\phi x}}{\partial \phi} + P_x \tag{41}$$

$$0 = \frac{1}{R}\frac{\partial^2 M_{x\phi}}{\partial x \partial \phi} + \frac{1}{R}\frac{\partial^2 M_{\phi x}}{\partial x \partial \phi} + \frac{1}{R^2}\frac{\partial^2 M_{\phi\phi}}{\partial \phi^2} + \frac{\partial^2 M_{xx}}{\partial x^2} - \frac{T_{\phi\phi}}{R} + Pr \tag{42}$$

3.1.3 Solution Process For The Bending Problem

The closed form solution can be obtained by substituting the equations into the equilibrium equations. The process of substitution into the equilibrium and compatibility equations is as follows:

1. Substitute only for $M\phi\phi$, $M\phi x$, Mxx, and $Mx\phi$ from the constitutive relations into the equilibrium equations. The equations will then be in terms of the membrane stress resultants $T\phi\phi$, $T\phi x$, Txx, and $Tx\phi$ and displacements u_o, v_o, and w_o.
2. Repeat step 1 for the compatibility equations. The compatibility equations will also be in terms of the membrane stress resultants $T\phi\phi$, $T\phi x$, Txx, and $Tx\phi$ and displacements u_o, v_o, and w_o.
3. Substitute for all stress resultants in terms of the strain-displacement functions. The result will be coupled partial differential equations in terms the displacements u_o, v_o, and w_o. The compatibility equations should be identically satisfied.
4. Solve the equilibrium equations in terms of the displacements.
5. The displacements can then be used to determine all the stresses in the elliptical cylinder.

At this time steps #1 through #3 have been completed. The problem lies in step #4. A method of solving the difficult equilibrium equation has not been established yet. If the equations are uncoupled a system of eighth order differential equations will be the result. The uncoupling yields fourth order partial differential equations for w_o, and second order partial differential equations for u_o, and v_o. As a result of the difficulty in solving higher order partial differential equations, Kozik recommends seeking a solution that does not involve uncoupling the equations.

3.2 BRESSE APPROACH

Pearson and Todhunter present M. Bresse's approach for solving the bending moment in an elliptical cylinder under internal pressure from his *Cours de Mecanique Appliquee, Premiere Partie* 1880. The case is entitled *Resistance d'une Chaudiere a Profil Failbliment Elliptique*, p. 326. Bresse treats the shell as a rod. He takes the product of the flexural rigidity and the change in curvature as equal to the bending moment. Bresse's formulation of the bending moment is as follows:

$$M = Ecl\frac{c^2}{12}\frac{\Delta\psi}{\partial s} \tag{43}$$

variables:

c = shell thickness

l = length of the cylinder

$\frac{\Delta\psi}{\partial s}$ = change due to strain in the angle between two tangents to the central line of the elliptical cross-section

Bresse's final formulation for the bending moment per unit length at any point on the elliptical profile includes the internal pressure and the location along the semi-major axis from the centroid of the ellipse.

$$M = \frac{1}{4}pe^2\left(2x^2 - a^2\right) \tag{44}$$

in which:

p = internal pressure

a = length of the semi-major axis

b = length of the semi-minor axis

x = the horizontal distance of any point on the profile measured from the center of the ellipse.

e = eccentricity of the ellipse

$$e = \sqrt{1 - \frac{b^2}{a^2}} \tag{45}$$

3.2.1 Bresse Bending Stress

When the geometry of the elliptical profile and internal pressure examined in this report are used in Bresse's equations the following plot of bending stress in one quadrant of the ellipse results. It is interesting to note that the positive and negative extreme values are equal in magnitude when Bresse's equations are used.

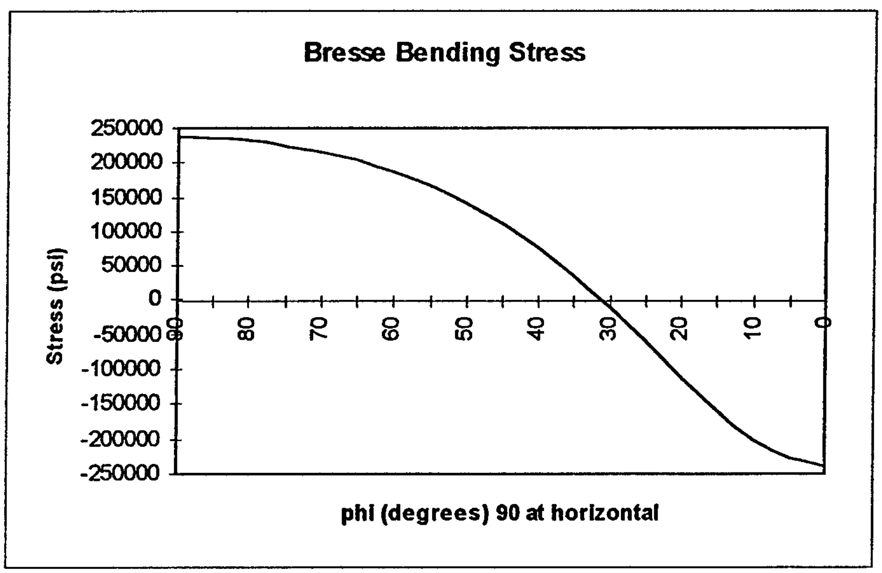

FIG. 13. Bresse Bending Stress

3.3 BROWN APPROACH

W. S. Brown's solution to the tension, shearing force and the bending moment of an elliptical cylinder under internal pressure were presented in *The Engineer*, September 23, 1904. His paper presented solutions to the bending moment in the form of graphs for ellipses having eccentricities which range between 0 and 1, and equations for ellipses of any eccentricity. Brown focused on the maximum values of tension, shearing and bending forces and their locations on the elliptical profile. With these values it is possible to locate longitudinal seams of cargo tanks at the location of zero bending moment.

Brown solves the bending problem using equilibrium of a section of the ellipse shown in Figure 15. A hoop of unit length is defined by points ABCD, Figure 14, where a and b represent the semi-major and semi-minor axes respectively. The section of the ellipse is AP with a uniform internal pressure of p.

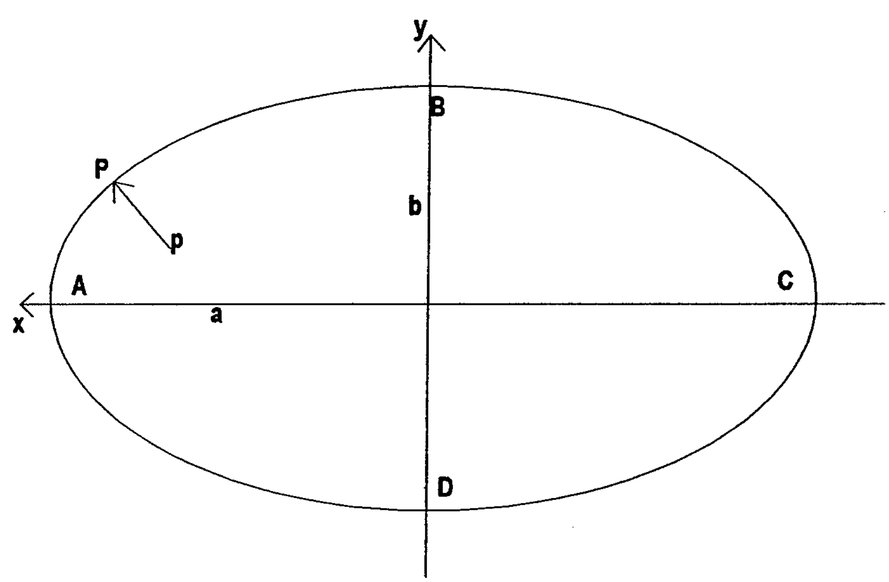

FIG. 14. Brown Elliptical Profile

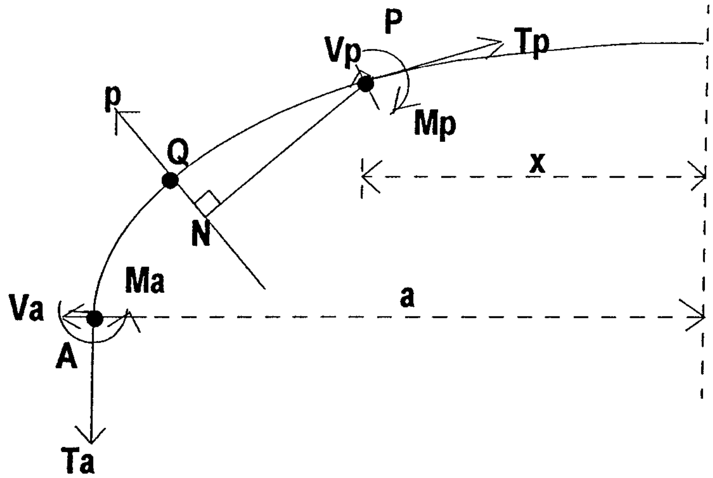

Fig. 15. Free Body Diagram of the Elliptical Profile

At P(x,y) there exists a bending moment M_p, and forces due to tension and shearing. By summation of vertical forces on the upper hemisphere of the ellipse it can be proved that $T_a = pa$. Summation of the moments about P for Fig. 15 equals:

$$M_P = M_A + \int_0^s p\overline{PN} \cdot ds - pa(a - x). \qquad (47)$$

$\overline{PN}$ is perpendicular from P to the force which is normal the the surface and acts on point Q. Using this argument, *pPNds* is the moment due to the internal pressure on the element *ds* at point Q about P. The integration performed with respect to s is the value o of s at A to the value of s at P. After completing the integration the following equation results:

$$M_P = M_A - \frac{p}{2}\left(a^2 - x^2 - y^2\right). \qquad (48)$$

The preceding equation allows the bending moment to be found at any point on the profile of the ellipse. It can also be shown that:

$$M_p = EI\left(\frac{1}{R} - \frac{1}{R_o}\right), \qquad (49)$$

where $\frac{1}{R_o}$ and $\frac{1}{R}$ are the curvatures of the ellipse before and after deformation. These values may also be written as:

$$\frac{1}{R_o} = \frac{\partial \psi_o}{\partial s} \tag{50}$$

$$\frac{1}{R} = \frac{\partial \psi}{\partial s}, \tag{51}$$

where ψ_o and ψ are the inclinations of the tangent at point P before and after strain, respectively. Due to the fact that the ellipse is a symmetrical shape ψ and ψ_o are equal at points A and C, and B and D before and after strain. Because of this, substitution for M_p into equation 21 and integrating between the limits yields the following equation:

$$o = M_a \bar{s} - \frac{p}{2}\int_o^s \left(a^2 - x^2 - y^2\right) ds. \tag{52}$$

In this equation x and y are the coordinates of point P, and $\bar{s}$ is the arc length of AB. Upon integration using the complete normal elliptic integrals of the first and second kind, K and E respectively, M_a can be obtained.

$$M_A = \frac{p}{6}\left(2a^2 - b^2 - b^2 \frac{K}{E}\right) \tag{53}$$

The moment at the semi-major axis will always give the largest bending moment. By substituting the moment at A into equation 48 the bending moment at any point on the elliptical profile can be determined. The moment at B will be the largest moment of the opposite sign of the moment at A. When solving for the moment at B equation 48 can be more compactly written as:

$$M_B = M_A - \frac{p}{2}\left(a^2 - b^2\right). \tag{54}$$

3.3.1 Brown Bending Stress

The graphical form the solution to the conditions of the elliptical profile and pressure examined in this report follow.

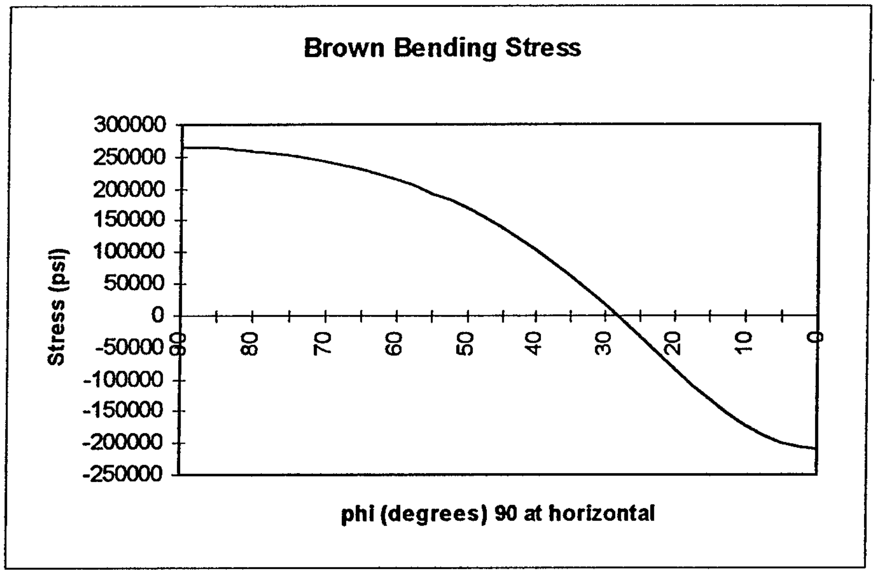

FIG. 16. Brown Bending Stress

As stated previously, the point of zero bending moment is of great concern in the design of elliptical cargo tanks. These points are were the seams must be placed to avoid the introduction of bending stresses. The zero bending points can be obtained by setting equation 48 to zero and rearranging to solve for the x and y coordinates.

$$x = a\sqrt{\frac{-M_B}{M_A - M_B}} \tag{55}$$

$$y = b\sqrt{\frac{M_A}{M_A - M_B}} \tag{56}$$

3.3.2 Brown Shear

The shearing force is also of concern in cargo tank design. Brown determined that it will have a maximum value of:

$$V_{\max} = \frac{p}{2}(2a - 2b)\,. \tag{57}$$

The shear will go to zero as the coordinates approach the axes of the ellipse. The maximum shear will occur at:

$$x = \sqrt{\frac{a^3}{a+b}} \tag{58}$$

$$y = \sqrt{\frac{b^3}{a+b}}\,. \tag{59}$$

3.4 TIMOSHENKO: ELLIPTIC RING

A simple approach for finding the bending moments in an elliptical ring is presented by S. Timoshenko in *Strength of Materials*, 1930. This simplification is thought to be a good approximation of an elliptical cylinder profile far removed from boundary conditions. Timoshenko provides coefficients which make it is possible to quickly evaluate the bending moments at the major and minor axis. These values prove to be very useful, due to the fact that these points will represent the maximum positive and negative bending moments on the shell. Timoshenko solves the problem using an elliptical quadrant shown in Fig. 17.

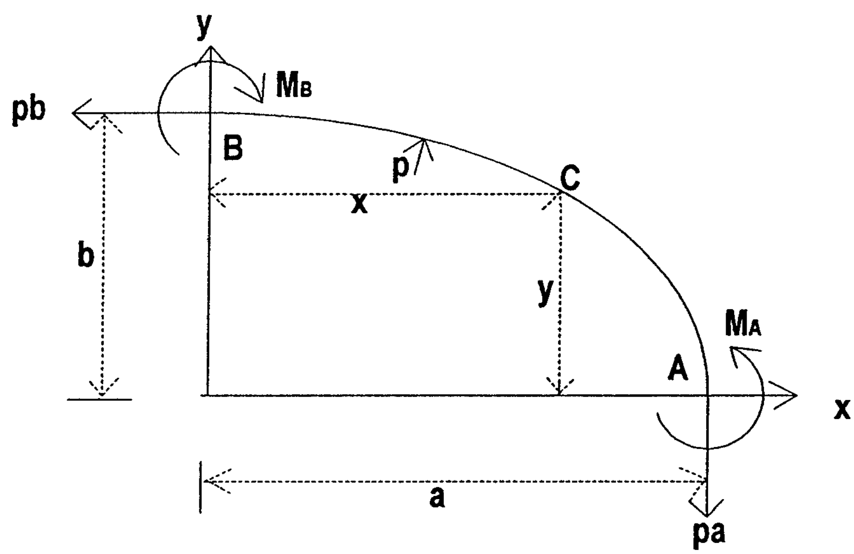

FIG. 17. Timoshenko Elliptical Quadrant

Since M_B represents the statically indeterminate bending moment at B, the bending moment at any arbitrary point on the quadrant can be found by the following equilibrium equation.

$$M_C = M_B - pb(b-y) + \frac{p(b-y)^2}{2} + \frac{px^2}{2} \tag{60}$$

$$M_C = M_B - \frac{pb^2}{2} + \frac{py^2}{2} + \frac{px^2}{2} \quad (61)$$

Timoshenko makes the problem very simple by computing coefficients that allow the bending moments to be found with simple equations and the chart of coefficients for beta, and gamma. The bending moments at the major and minor axes can be quickly calculated with only the internal pressure, the ratio of the minor to major axes, and Timoshenko's table of coefficients.

Bending moment at the major axis:

$$M_A = pb^2\gamma\,. \quad (62)$$

Bending moment at the minor axis:

$$M_B = -pb^2\beta\,. \quad (63)$$

Using the geometry of the ellipse and the table below to obtain beta and gamma the bending moments is easily solved.

b/a	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3
β	0	0.057	0.133	0.237	0.391	0.629	1.049	1.927
γ	0	0.06	0.148	0.283	0.498	0.87	1.576	3.128

Table 1. Timoshenko Constants for Elliptic Ring Bending Moments

Timoshenko's method is expedient and gives the maximum and minimum bending moments without in-depth calculations. The solution for the parameters in this report are as follows:

$$\frac{b}{a} = 0.602 \approx 0.6$$

From Table 1, $\beta = 0.391$ and $\gamma = 0.498$. Now by applying equations 62 and 63 the extreme positive and negative bending moments are obtained.

$$M_A = 10psi \cdot (23.75in)^2 \cdot 0.498 = \underline{\underline{2{,}809lbs}}$$

$$M_B = -10psi \cdot (23.75in)^2 \cdot 0.391 = \underline{\underline{-2{,}205lbs}}$$

With the moment at B and the x-y coordinates, the bending moment at any point on the ellipse can be found by solving equation 61. The equations for the x and y coordinates are given below, where ϕ is the angle which relates the tangent to point C and the horizontal to the minor axis. These are the same equations derived by Flugge from section 2.5.

$$x = \frac{a^2 \sin\phi}{(a^2 \sin^2\phi + b^2 \cos^2\phi)^{1/2}} \tag{64}$$

$$y = \frac{b^2 \cos\phi}{(a^2 \sin^2\phi + b^2 \cos^2\phi)^{1/2}}. \tag{65}$$

3.4.1 Timoshenko Bending Stress

The resulting bending stress in an elliptical ring using Timoshenko's derivation is shown in Fig. 15.

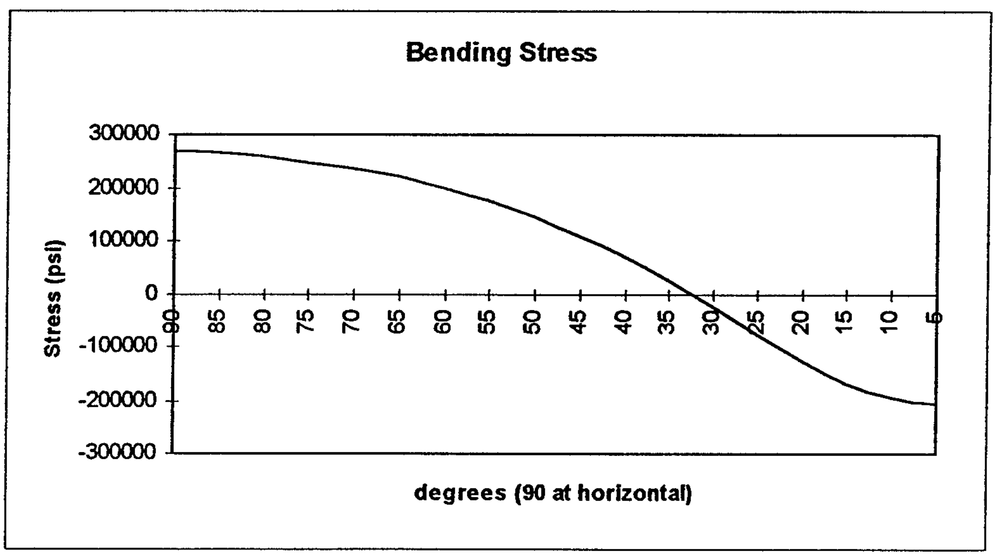

FIG. 18. Timoshenko Bending Stress

3.5 Holland, Lalor, and Walsh Elliptical Cylinder

In their paper titled, "Principal Displacements in a Pressurized Elliptical Cylinder: Theoretical Predictions with Experimental Verification By Laser Interferometry", Holland, Lalor and Walsh develop equations to ultimately predict displacements. As a necessity to finding the displacements, they had to developed equations for bending moments (M) and hoop force (T). These two values are both required to find the total stresses in the cylinder wall.

The authors make extensive use of complete elliptic integrals of the first kind (F) and complete elliptic integrals of the second kind (K). Five-place tables of complete elliptic integrals were used to find the values for F and K. Phi is a function of the geometry of the elliptical profile.

$$\phi = Cos^{-1}\left(\frac{b}{a}\right) \tag{66}$$

The value of phi, which differs from the phi used in the previous sections, is used to find F and K from LeGengre's elliptic integral tables. The constants A and B are used to calculate the bending moment (M). Figure 19 shows the geometry and variables used by Holland, Lalor, and Walsh to develop their equations. An element of the shell having a length of *ds* is located at *A* and related to the eccentric angle ε. The inscribed circle within the ellipse has a radius equal to the semi-minor axis *b*. The infinitesimal element ds is subjected to the forces of tension T and bending moment M. The element increases in length from the tensile force by the amount $\frac{T}{tE}$, where t is the shell thickness and E Young's Modulus. The element also rotates due to the bending moment by an amount of

$\frac{M}{EI}ds$. The authors take EI as the flexural rigidity of a plate, not as a rod as was done previously by Bresse and Brown.

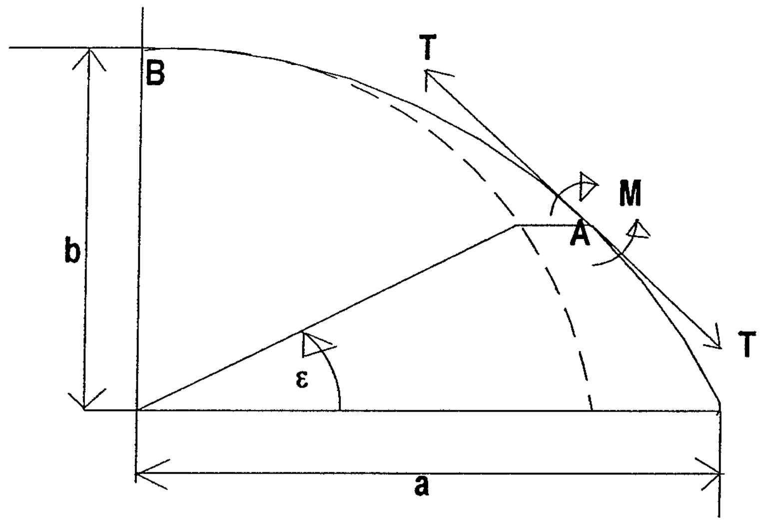

FIG. 19. Holland, Lalor & Walsh Cylinder Loading

$$A = \frac{pb^2}{6}\left(1 + 2Tan^2\phi - \frac{K}{F}\right) \tag{67}$$

$$B = \frac{pb^2}{2}Tan^2\phi \tag{68}$$

$$M = A - B\sin^2\varepsilon \tag{69}$$

$$T = \frac{pb}{\sqrt{1 - Sin^2\phi Cos^2\varepsilon}} \tag{70}$$

The identical conditions and geometry's are used in this section as in the previous sections of this report. They are:

p = 10 psi.

a = 39.4 in.

b = 23.75 in.

3.5.1 Holland, Lalor, and Walsh Bending Stress

Using these parameters, the graphical representation of the bending moments are shown below in figure 20.

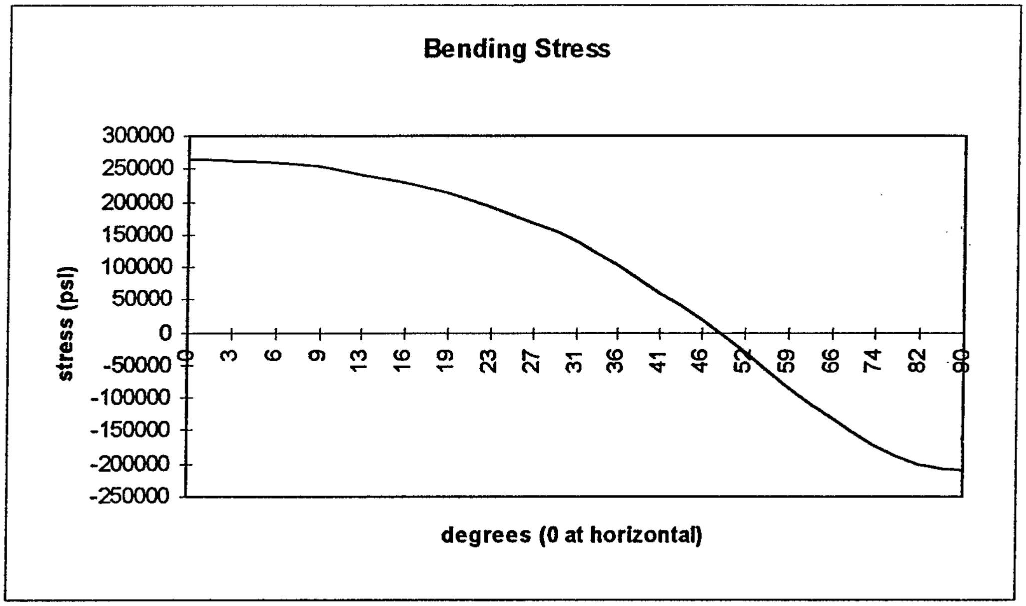

FIG. 20. Holland, Lalor, and Walsh Bending Stresses

3.5.2 Holland, Lalor, and Walsh Hoop Stress

Unlike the previous authors Holland, Lalor, and Walsh provide the means to determine the hoop forces throughout the elliptical profile. A graph of the hoop forces is provided in Fig. 21. These hoop forces when combined with the bending moments give the ability to determine the total stresses which result in the shell.

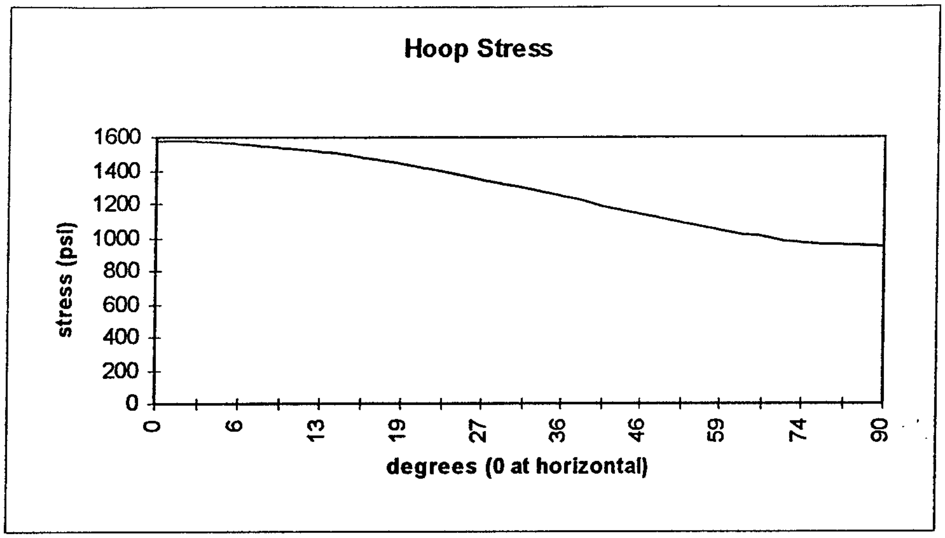

FIG. 21. Holland, Lalor, and Walsh Hoop Stresses

3.5.3 Holland, Lalor, and Walsh Total Stress

Once T and M are found using equations 69 and 70, it is possible to find the total stress at any point on the shell of the elliptical cylinder using equation 1.

$$\sigma_\phi = \left(\frac{N_\phi}{t}\right) + \left(\frac{6M\phi}{t^2}\right). \tag{1}$$

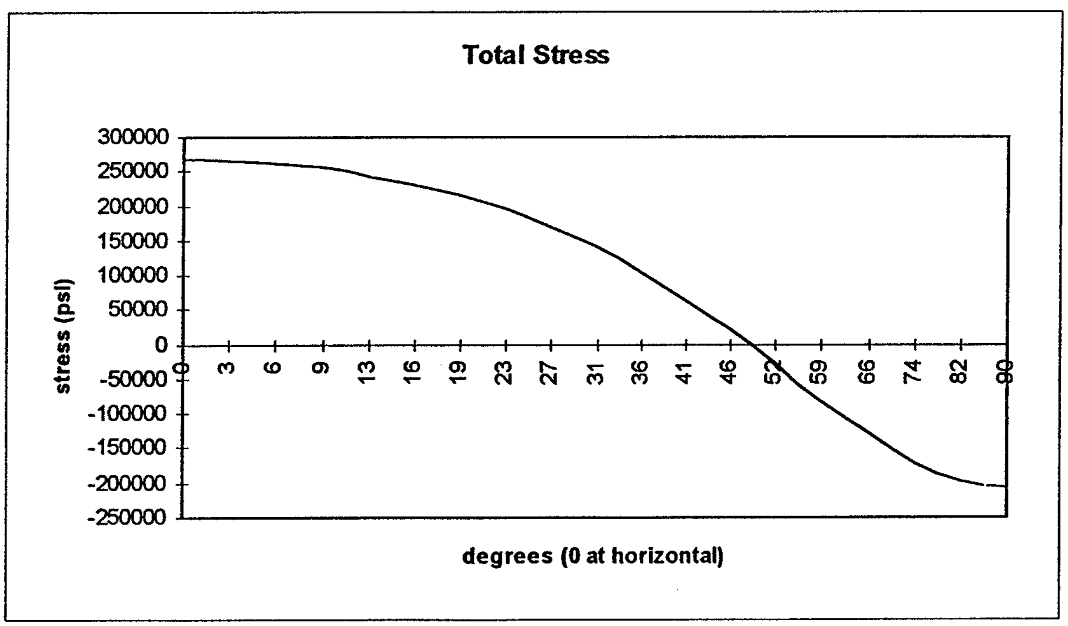

FIG. 22. Holland, Lalor, and Walsh Total Stresses

3.6 KOZIK: EQUILIBRIUM SOLUTION TO THE ELLIPTICAL CYLINDER

Dr. Thomas J. Kozik of Texas A&M University developed a solution for the shearing, hoop, and bending forces that are known to be present in a non-circular cylinder. The equations presented in section were derived using the principles of equilibrium specifically for use with elliptical cylinders under internal pressure. The derivation is quite lengthy, however only the most critical steps are presented for brevity. The basis for the solution is the infinitesimal surface element shown in Fig. 23.

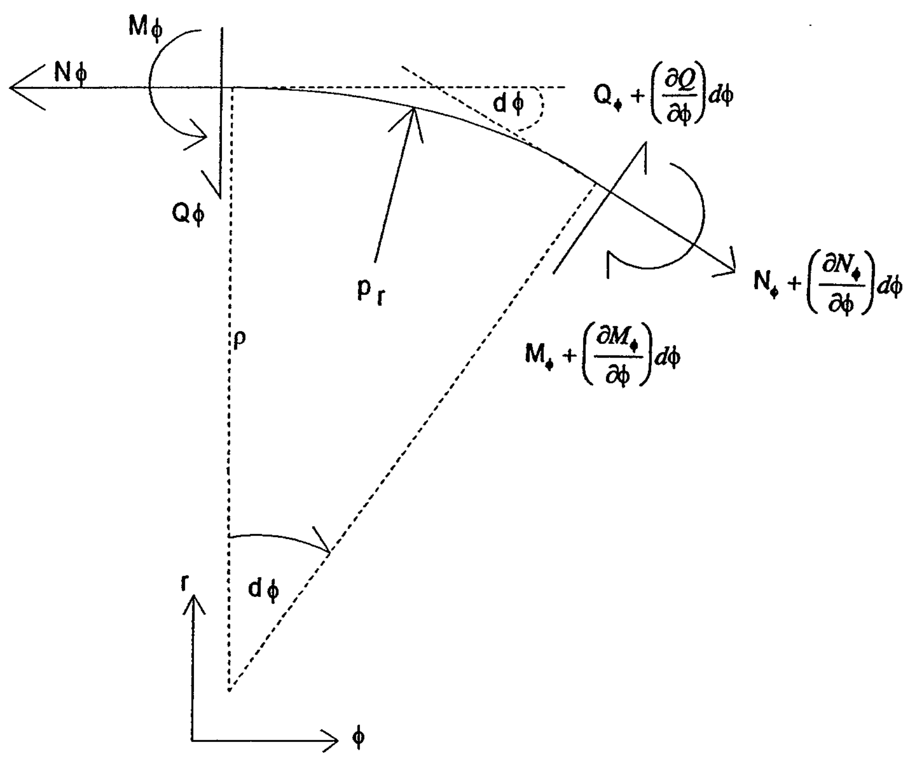

FIG. 23. Kozik Surface Element

The derivation begins with the summation of forces in the radial direction or the direction normal to the surface of the shell, the ϕ-direction, and by summation of moments. The results follow:

Summing forces in the radial direction provides the following equilibrium equation:

$$-Q_\phi + \left(Q_\phi + \frac{\partial Q}{\partial \phi} d\phi\right) + p\rho d\phi - \left(N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi\right) d\phi = 0 \tag{71}$$

Which reduced to

$$\frac{\partial Q}{\partial \phi} + p\rho - N\phi = 0 \tag{72}$$

Next the forces are summed on the ϕ-direction.

$$-N_\phi + \left(N_\phi + \frac{\partial N}{\partial \phi} d\phi\right) + \left(Q_\phi + \frac{\partial Q_\phi}{\partial \phi} d\phi\right) d\phi = 0 \tag{73}$$

This equation is reduced to

$$\frac{\partial N_\phi}{\partial \phi} + Q_\phi = 0. \tag{74}$$

Finally, the summation of moments yields:

$$-M_\phi + \left(M_\phi + \frac{\partial M}{\partial \phi} d\phi\right) - \left(Q_\phi + \frac{\partial Q_\phi}{\partial \phi} d\phi\right) \rho d\phi = 0 \tag{75}$$

Which after elimination becomes:

$$\frac{\partial M_\phi}{\partial \phi} - \rho Q_\phi = 0 \tag{76}$$

where,

$$\rho = \frac{a^2 b^2}{\left[a^2 \sin^2 \phi + b^2 \cos^2 \phi\right]^{3/2}}. \tag{77}$$

The equations for the hoop force can be written as

$$N_\phi = \frac{\partial Q}{\partial \phi} + p\rho \tag{78}$$

$$\frac{\partial N_\phi}{\partial \phi} = \frac{\partial^2 Q}{\partial \phi^2} + p\frac{\partial p}{\partial \phi} \quad (79)$$

Applying equation (79) to equation (74) yields:

$$\frac{\partial^2 Q}{\partial \phi^2} + Q_\phi = -p\frac{\partial p}{\partial \phi} \quad (80)$$

With these relationships established it is necessary to seek further relationships using Fig. 24. The following figure is a free body diagram of an elliptical cylinder which has been cut between the major and minor axes.

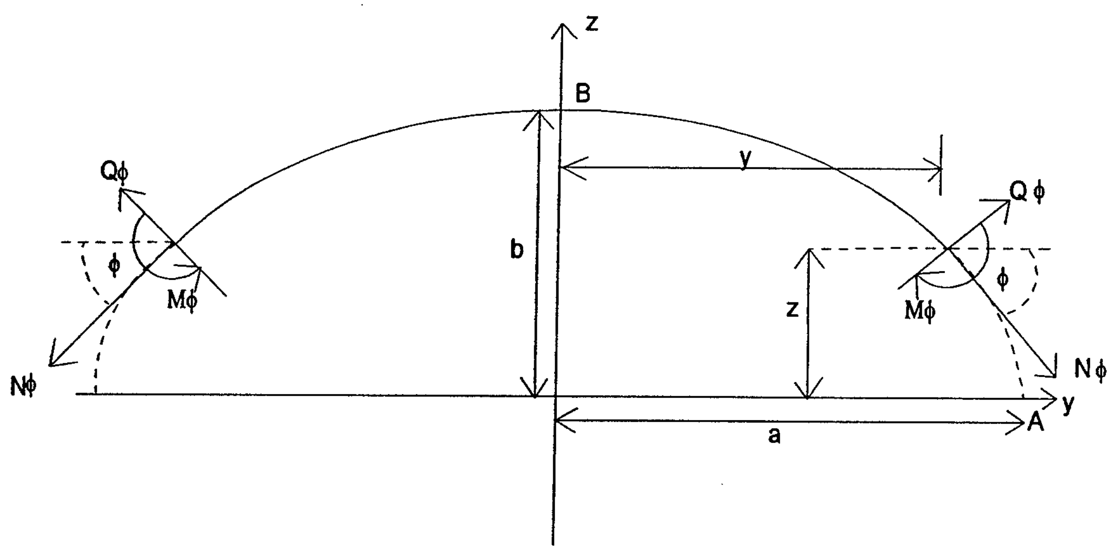

FIG. 24. Kozik Equilibrium Profile

Using summation of forces in the z-direction the following equation was constructed:

$$N_\phi Sin\phi - Q_\phi Cos\phi = py. \quad (81)$$

The distance from the centroid of the ellipse along the major-axis to the cut in the shell is y, and the equation is:

$$y = \frac{a^2 Sin\phi}{\sqrt{a^2 Sin^2\phi + b^2 Cos\phi}} = \frac{a^2 Tan\phi}{\sqrt{b^2 + a^2 Tan^2\phi}}. \quad (82)$$

Combining equations (81) and (82) gives:

$$N_\phi Sin\phi - Q_\phi Cos\phi = \frac{pa^2 Tan\phi}{\sqrt{b^2 + a^2 Tan^2\phi}} \tag{83}$$

By combining the equations established to this point the succeeding two equations arise.

$$N_\phi - \frac{\partial Q}{\partial \phi} = \frac{pa^2 b^2}{\left[a^2 \sin^2\phi + b^2 \cos^2\phi\right]^{3/2}} \tag{84}$$

$$N_\phi Sin\phi - Q_\phi Cos\phi = \frac{pa^2 Sin\phi}{\left[a^2 \sin^2\phi + b^2 \cos^2\phi\right]^{1/2}} \tag{85}$$

The result of combining equations 74 and 84 is:

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = \frac{pa^2 b^2}{\left(a^2 Sin^2\phi + b^2 Cos^2\phi\right)^{3/2}} \tag{86}$$

Likewise, equations 46 and 57 may be combined to form:

$$N_\phi Sin\phi + \frac{\partial N_\phi}{\partial \phi} Cos\phi = \frac{pa^2 Sin\phi}{\left(a^2 Sin^2\phi + b^2 Cos^2\phi\right)^{1/2}} \tag{87}$$

In order to solve these equations it is necessary to find the homogeneous and particular solutions, uncouple the equations, and finally integrate to get the final answer. The solution is detailed and lengthy, therefore the intermediate steps have been left out and the solution for N_ϕ is:

$$\underline{\underline{N_\phi = p\sqrt{a^2 Sin^2\phi + b^2 Cos^2\phi}}}, \tag{88}$$

and the solution for the shearing force is:

$$\underline{\underline{Q_\phi = -\frac{p\left(a^2 - b^2\right) Cos\phi Sin\phi}{\sqrt{a^2 Sin^2\phi + b^2 Cos^2\phi}}}} \tag{89}$$

Much of the work previously accomplished can be applied to finding the solution to the bending moment. The steps to solving the bending moment are given below.

$$\rho Q_\phi = \frac{\partial M_\phi}{\partial \phi} \tag{90}$$

$$Q_\phi = -\frac{\partial N_\phi}{\partial \phi} \tag{91}$$

$$\frac{\partial M_\phi}{\partial \phi} = -\rho \frac{\partial N_\phi}{\partial \phi} \tag{92}$$

$$\frac{\partial N_\phi}{\partial \phi} = \frac{p\left(a^2 - b^2\right) Cos\phi Sin\phi}{\sqrt{a^2 Sin^2\phi + b^2 Cos^2\phi}} \tag{93}$$

$$\frac{\partial M_\phi}{\partial \phi} = \frac{pa^2b^2\left(a^2 - b^2\right) Cos\phi Sin\phi}{\left(a^2 Sin^2\phi + b^2 Cos^2\phi\right)^2} \tag{94}$$

Using trigonometric identities to simplify equation 94 followed by integration the expression for the bending moment is obtained.

$$M\phi = -\frac{pa^2b^2}{\left\{\left(a^2 + b^2\right) - \left(a^2 - b^2\right) Cos2\phi\right\}} + C \tag{95}$$

The constant of integration was initially thought to be zero. After further study it was discovered that the constant was critical in determining the bending moment using this solution technique. For Kozik's method to work properly a bending moment that is known to be correct at either the major or minor axis must first be calculated from one of the other techniques previously presented in this report. For ease of calculation, it is suggested to use the method derived by Timoshenko for the elliptic ring for the initial bending moment at the major axis.

$$C = M_A{}^T + \frac{pb^2}{2} \tag{96}$$

where, $M_A{}^T$ is the Timoshenko bending moment at the major axis.

3.6.1 Kozik Hoop, Bending, and Total Stress

The graphical solutions to hoop force, shearing force, and bending moment are given in Figs. 25 - 28.

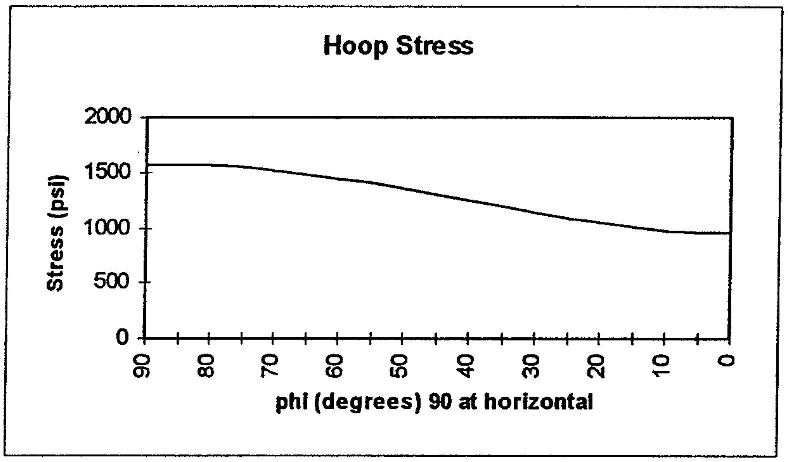

FIG. 25. Kozik Hoop Stress

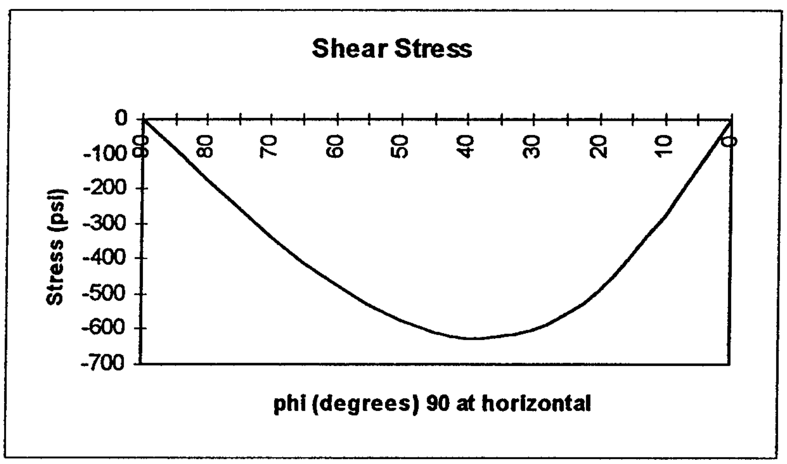

FIG. 26 Kozik Shear Stress

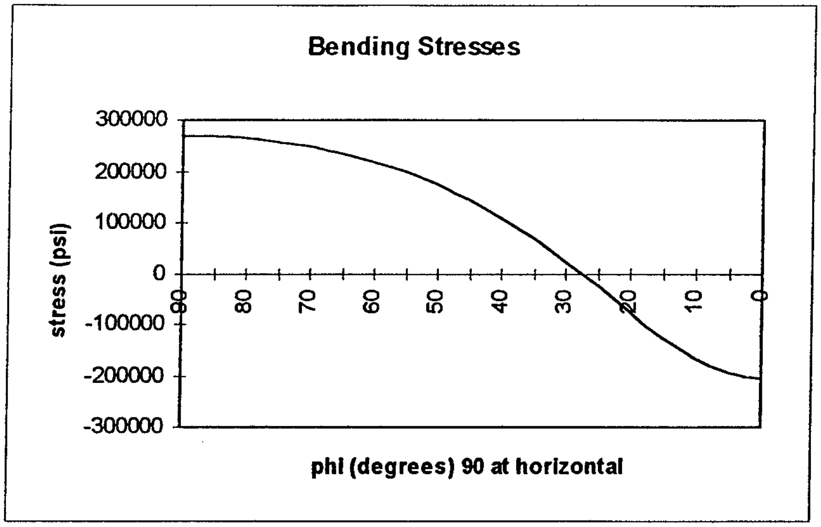

FIG. 27. Kozik Bending Stress

Now that both the hoop force and the bending moment have been calculated it is possible to find the bending stresses created in the shell according to the Kozik derivation.

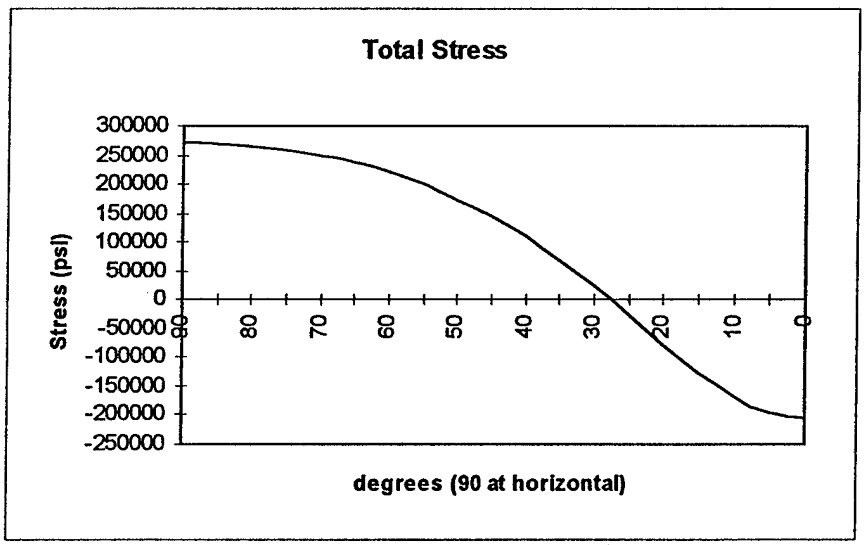

FIG. 28. Kozik Total Stress

4. FINITE ELEMENT ANALYSIS

4.1 General

All Finite Element Models (FEM's) for use in this report were constructed using the principle of symmetry. Since all four quarters of the ellipse are identical, only one quarter of the ellipse was modeled. Figure 29 shows a typical model and the constraints. The upper right hand quadrant was used in all applications. In order to achieve a realistic model of the complete profile, the nodes at the minor and major axes were only allowed to translate vertically and horizontally respectively. These nodes were also fixed in rotation to simulate their connection to the other quadrants of the ellipse. All other nodes are constrained to prevent longitudinal translation and in-plane rotations. The rotation constraint prevented bowing of the elements in the longitudinal direction of the shell between the nodes.

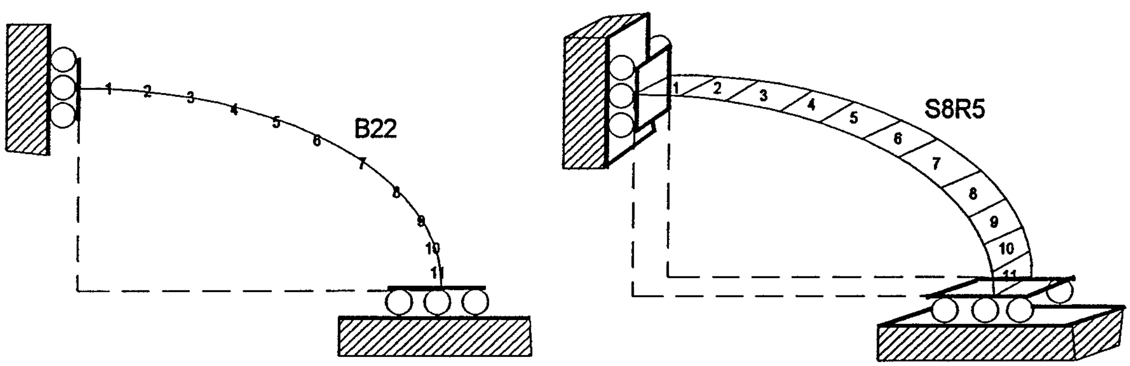

FIG 29. FEM Model and Constraints

4.2 Software

4.2.1 Microsoft WORD and EXCEL

EXCEL was used to create the nodes used to generate the circular and elliptical profiles. The equations for the distances in the x and y directions were generated using

the angle phi (ϕ) which progresses from zero degrees at the minor axis to 90 degrees at the right side major axis. Equations 64 and 65 from section 3.4 were used to generate the x-y coordinates. WORD was used to edit the values imported from EXCEL and edit the input files for ABAQUS.

4.2.2 ABAQUS and ABAQUS POST

After the files were finalized, ABAQUS was used to analyze the input. The output from ABAQUS that is of primary use in this study are deflections, section forces, and section moments. ABAQUS POST was used to graphically view the results generated by ABAQUS. ABAQUS POST allowed the deflected shape and color plots of the forces to be viewed in color. Black and white, as well as, color outputs used in analysis were created using ABAQUS POST.

4.3 Models

All loads were applied to simulate a uniform internal pressure acting on the inner surface of the elements. Loading was applied such that it always remains normal to the surface of the element. A "P2" loading was used with beam elements and "P" loading was used with all shell elements. Both loading conditions meet the critical criteria of remaining normal with respect to the inner surface of the elements during displacement.

In all cases a circular section was modeled first to gain confidence that the FEM was functioning properly. The circle was chosen because there is great confidence in the closed form solutions for the hoop stress and the radial deflection. Only after confidence

in the circular FEM had been established, were the more complex elliptical models analyzed.

Both beam and shell elements were used in the finite element models. Beam elements used were three-nodded quadratic B22 elements. Eight-nodded S8R5 elements were used in the shell models. These elements were chosen primarily because they are quadratic, like the geometry of an ellipse. This allows the true geometric profile to be closely modeled.

4.4 Finite Element Analysis

The models were originally analyzed using an internal pressure of 10 psi. This caused problems when the data was compared to the expected results from the analytical solutions previously presented in this report. Very large displacements occurred in the FEM at the major and minor axes. This made the results of the FEM and the analytical solutions incompatible. It is clear from the results presented earlier that the authors of the closed form solutions were utilizing small deflection theory. This can be seen by examining the calculations for the hoop force at the major and minor axes. Regardless of the load applied to the shell, the hoop force at these points is simply the internal pressure multiplied by the length of the axis. Clearly, the authors were only considering very small deflections. In order to compare the results of the FEM to the analytical solutions, it was necessary to apply loads to the FEM that provided deflections that were less than or equal to one-half the thickness of the shell. The load was reduced to 0.01 psi, which is a decrease of three orders of magnitude, and the results were recalculated. The deflections obtained using this load were well within the limits of small deflection theory. The

following graphs shows the results of the FEM using the reduced load compared to the results using the equations developed by Kozik.

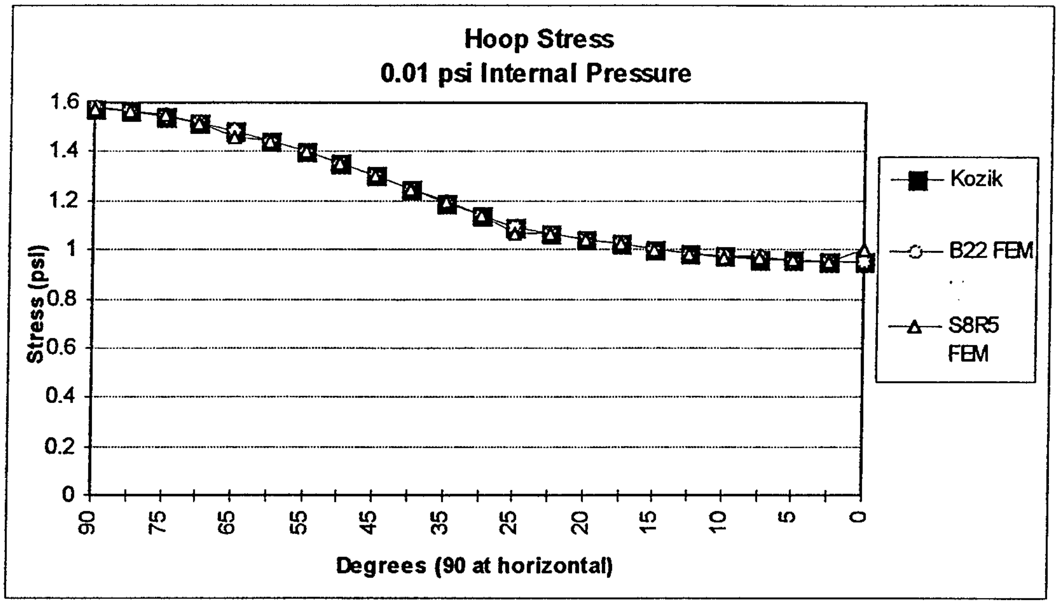

FIG. 30. FEM's vs. Kozik Hoop Stress

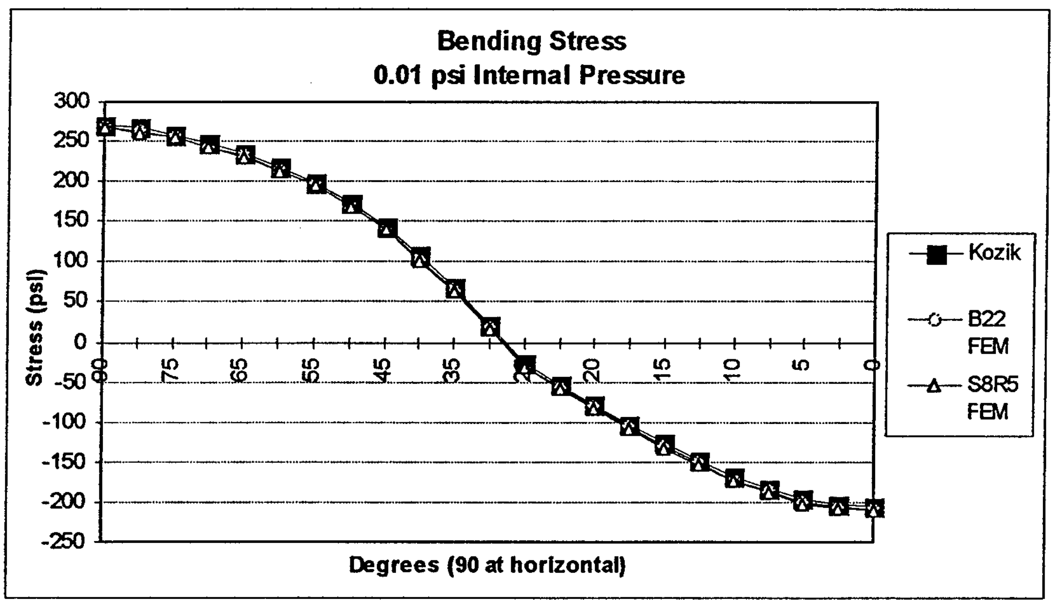

FIG. 31. FEM's vs. Kozik Bending Stress

The graphical representation of the data closely resembles that obtained from analytical solutions. This gives confidence that these FEM's can be used to determine the hoop stress and bending stress at any point in an elliptical cargo tank cross section as long

as small deflection theory is applied. The ABAQUS input files used for the beam and the shell FEMs are shown Appendix A and Appendix B, respectively.

5. COMPARISON OF ANALYTICAL METHODS

5.1 Hoop Stresses

Out of the six methods studied in this report only three provided a solution to the hoop forces that occur in elliptical cylinders. Flugge, Kozik and Holland, Lalor and Walsh give equations to calculate the hoop stress.

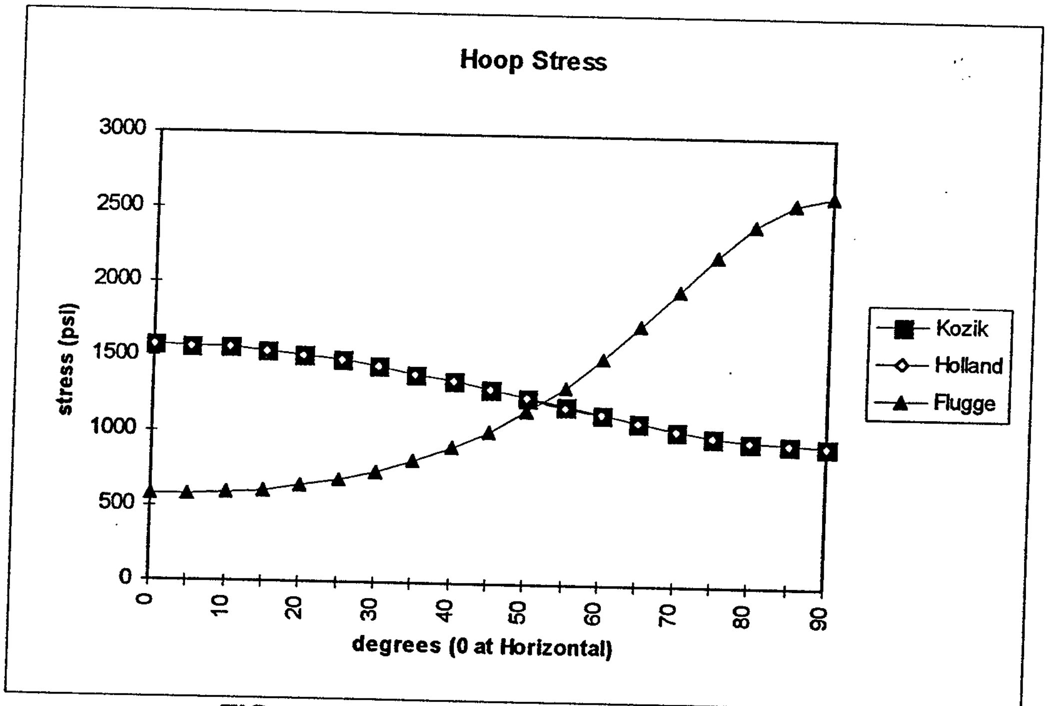

FIG. 32. Comparison of Hoop Stresses

Figure 32 shows that the formulations of Kozik and Holland, Lalor, and Walsh yield nearly identical results. The angle used by Holland, Lalor and Walsh is not the same as that utilized by Kozik. Holland, Lalor and Walsh relate the angle to an inscribed circle with a radius equal to the semi-minor axis (see section 3.5, Fig. 19). Kozik uses the angle formed by the tangent to a point on the shell profile with the horizontal (see section 3.6, Fig. 24). In order to achieve the results shown in the graph the angles had to be

manipulated. This was accomplished by relating the y-coordinates of the two methods. Recall equation 38 which gives the y-coordinate for any angle phi for an ellipse.

$$y = \frac{b^2 \cos^2 \phi}{(a^2 \sin^2 \phi + b \cos^2 \phi)^{1/2}} . \tag{65}$$

Holland, Lalor and Walsh use an inscribed circle and the angle epsilon from Fig. 16. The inscribed circle has the radius of the semi minor axis b. With these three values the age epsilon can be related to the angle phi by the following equation.

$$\varepsilon = Sin^{-1}\left(\frac{y}{b}\right) \tag{97}$$

The hoop force results provided when the equations derived by Flugge give entirely different values than the previous two authors. Upon closer examination, it is clear that Flugge's results are fundamentally incorrect. This can be proved by cutting the shell at the major and minor axes as shown is Fig. 33.

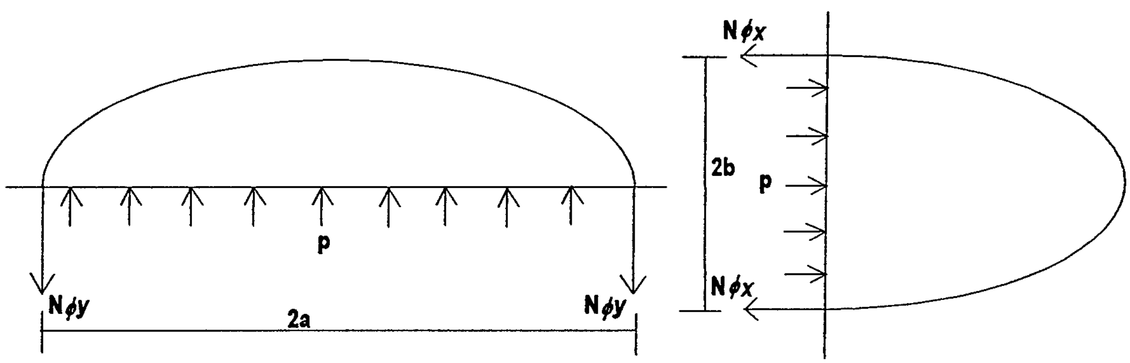

FIG. 33. Vertically and Horizontally Cut Elliptic Shell

By summing the forces vertically and horizontally the resulting equations are:

$$\Sigma F_y = N_{\phi y} = pa \tag{98}$$

$$\Sigma F_x = N_{\phi x} = pb \tag{99}$$

When these values are computed, the Kozik and Holland, Lalor, and Walsh answers are obtained identically. The result of the Flugge's formulation differs significantly. Flugge relates the hoop force to the radius of curvature multiplied by the pressure. This is why his values are so erroneous. With this simple proof using summation of forces it becomes clear that the equations developed by Flugge will not be useful in determining both the membrane stresses in cargo tank of elliptical cross sections.

5.2 Bending Stresses

Five of the six authors studied within this report, the only exception being Flugge, provide methods to determine bending moments. Four of the Five methods provide nearly identical results. Bresse's approach is very simple and has a curve that has the same shape as the others, but it is shifted down. His method should be questioned not only because it is the only one that is different, but also because the bending moments at both axes are equal. The comparison of all bending stresses is graphically shown in Fig. 34.

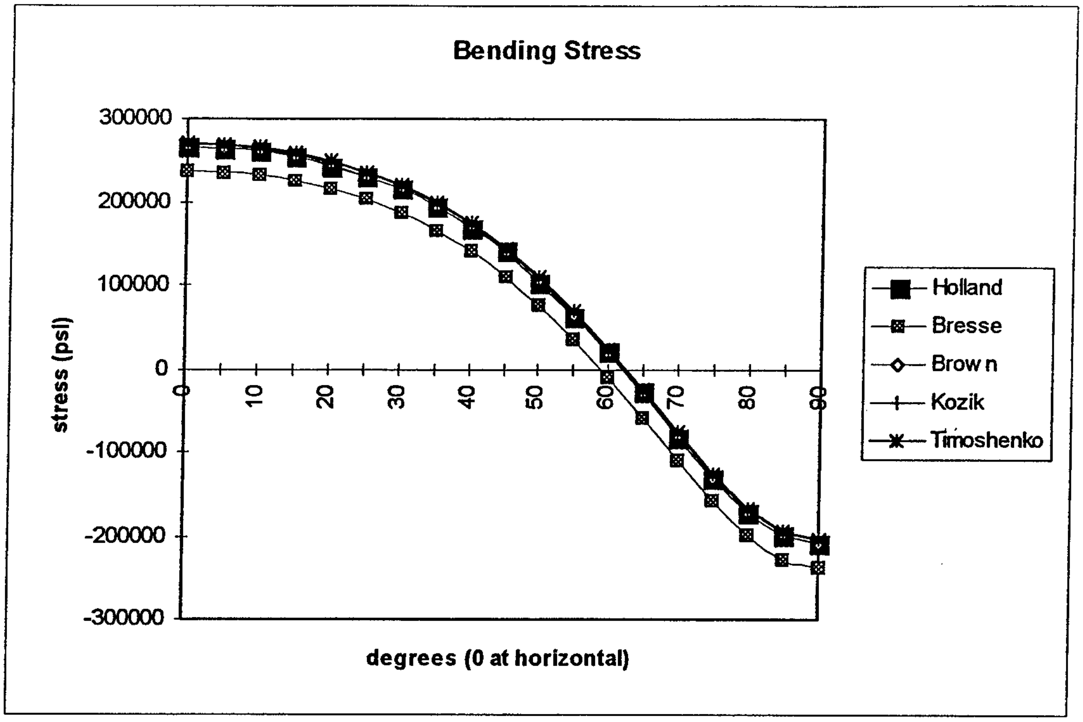

FIG. 34. Comparison of Bending Stresses

5.3 Total Stresses

When ultimately determining the total stresses, only two authors provide a complete set of equations that give the ability to find the total stress in the shell. The total stresses given by Kozik and Holland, Lalor and Walsh are unique because they are the only authors who provide solutions to the hoop forces. The two methods provide nearly identical results. This should be expected since the hoop and bending stresses previously shown are virtually the same.

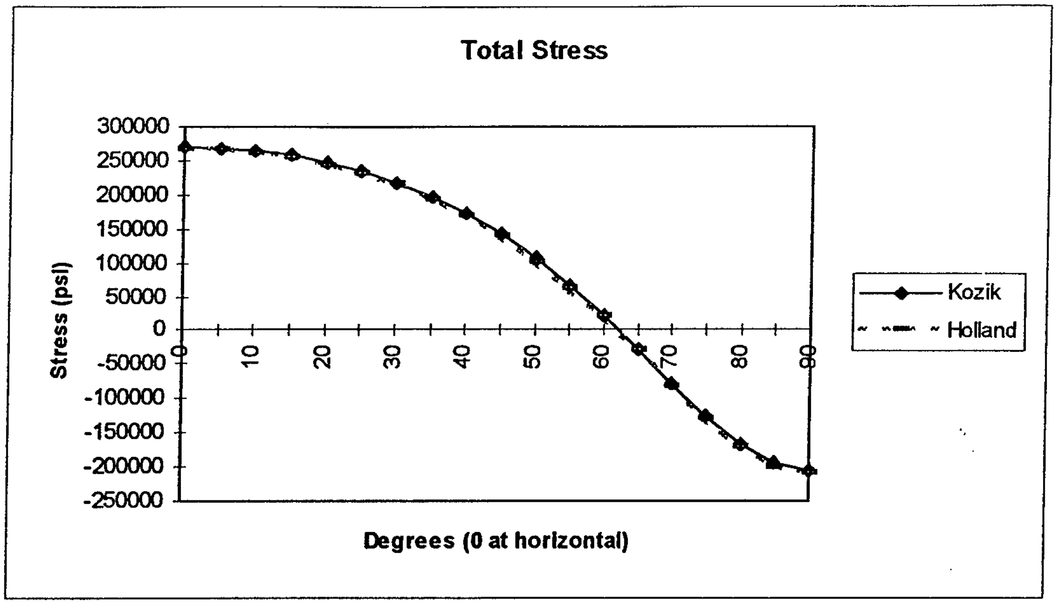

FIG. 35. Comparison of Total Stresses

6. CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

The solution provided by membrane theory is of little use when applied to any shape other than a circular cylinder or a sphere. Pure membrane theory neglects any transverse shearing forces, and bending moments which are known to occur when pressure is applied to an elliptical profile. The membrane theory based method presented by W. Flugge obviously provided no means by which to calculated bending moments since they are neglected. However, he provided a solution to the hoop force. As was shown in the previous comparison section, Flugge's procedure fails to provide accurate results for the hoop stress. This failure of pure membrane theory was proven by a comparison against simple static equilibrium.

The geometry specific equations developed for the elliptical cylinder using bending theory provide a way to determine the bending stresses experienced in the shell. The solution will involve solving a complex system of coupled partial differential equations. A technique for solving the equilibrium equations in terms of the strain-displacements functions u_o, v_o, and w_o has not been accomplished to this date.

The methods which account for bending in an elliptical shell presented by Brown, Timoshenko, Kozik, and Holland, Lalor, and Walsh all produce results that are nearly identical. This similarity of answers is not only present among the analytic solutions, but also in the Finite Element Models. The reliability of these results gives confidence in their accuracy. The analytic techniques used varying approaches and all yielded the same answer.

6.2 Recommended Solution Procedure

Since the methods previous described all give the same results, the recommended solution procedure consists of the methods which are the easiest to apply. The following outline provides the equations which provide the best methods to find the total stress in an internally pressurized elliptical cargo tank.

1. Hoop Force: Kozik

$$N_\phi = p\sqrt{a^2 Sin^2\phi + b^2 Cos^2\phi}$$

Kozik's equation needs only the values for internal pressure, the ellipse geometry, and the angle phi.

2. Bending Moment: Timoshenko

b/a	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3
β	0	0.057	0.133	0.237	0.391	0.629	1.049	1.927
γ	0	0.06	0.148	0.283	0.498	0.87	1.576	3.128

Bending moment at the major axis:

$$M_A = pb^2\gamma .$$

Bending moment at the minor axis:

$$M_B = -pb^2\beta .$$

Bending moment at any point:

$$M_C = M_B - \frac{pb^2}{2} + \frac{py^2}{2} + \frac{px^2}{2}$$

Timoshenko's coefficient table and set of equations allows the moments to be calculated at the major and minor-axes with only the ellipse geometry and internal pressure. To calculate the bending moment at any point between the axes, the only

additional values needed are the x-y coordinate of the point. The angle phi may be related to the x-y coordinate by the following two equations.

$$\mathrm{x} = \frac{a^2 \sin\phi}{(a^2 \sin^2\phi + b\cos^2\phi)^{1/2}}$$

$$\mathrm{y} = \frac{b^2 \cos\phi}{(a^2 \sin^2\phi + b\cos^2\phi)^{1/2}}.$$

3. X-Y Coordinate of Zero Bending Moment: Brown

$$x = a\sqrt{\frac{-M_B}{M_A - M_B}}$$

$$y = b\sqrt{\frac{M_A}{M_A - M_B}}$$

The moments at the major and minor axes found using Timoshenko's ethod can be applied to Brown's equations to find the location of zero bending moment. This location is critical to joint placement in cargo tanks.

4. Total Stress:

$$\sigma_\phi = \left(\frac{N_\phi}{t}\right) + \left(\frac{6M\phi}{t^2}\right).$$

The total stress in the shell can be found by utilizing Kozik's hoop force, Timoshenko's bending moment, and the thickness of the shell.

This sequence of equations provides a complete solution for determining the hoop force, bending moment, and total stress in any elliptical cargo tank subjected to internal pressure. Section 5 of this report proved that results of Brown, Timoshenko, Kozik, and

Holland, Lalor, and Walsh are very close to being equal and may be utilized in various combinations due to the similitude of their results.

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APPENDIX A

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